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ORIGINAL PAPER

Fundamental notions for the second generation Fukui project and a prototypal problem of the normed repeat space and its super spaces

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Abstract Fukui's DNA problem is a long-range target of the First and Second Generation Fukui Project, whose underlying motive has been to cultivate a new interdisciplinary region between chemistry and mathematics. The normed repeat space $\mathscr{X}_r(q, d, p)$ and its super space $\mathscr{X}_B(q, d, p)$ are special Banach algebras which are fundamental in the second generation Fukui project. In the present article, keeping Fukui's DNA problem in mind, we formulate and solve a prototypal problem (problem I) of the normed repeat space $\mathscr{X}_r(q, d, p)$ and its super space $\mathscr{X}_B(q, d, p)$. The present article provides also three "Challenging Problems" II–IV. These are called parallel problems in view of the structural parallelism observed when compared with the original problem I. The challenging problems, which would allow multiple approaches, are specially designed for those who are interested in interdisciplinary investigations.

Keywords Fukui conjecture · Repeat space theory (RST) · Additivity problems · Asymptotic linearity theorem (ALT) · Banach algebras

1 Introduction

Fukui's DNA problem is a long-range target of the First (cf. [1]) and Second Generation Fukui Project, whose underlying motive has been to cultivate a new interdisciplinary region between chemistry and mathematics, especially for tackling what we call *globally-pertaining-type problems*, or, for short, *g-type problems* [2];

This article is dedicated to the memory of the late Professors Kenichi Fukui and Haruo Shingu.

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these constitute physicochemical problems for whose solutions global mathematical contextualization is essential. "Can the conductivity and other properties of a single-walled carbon nanotube be analyzed in the setting of a $*$-algebra equipped with a complete metric?" This metric problem is fundamental to proceed towards the solution of Fukui's DNA problem. In the recent publication [3] by the present author, this metric problem was affirmatively solved and the new notion of normed repeat space $\mathscr{X}_r(q, d, p)$ was introduced. The normed repeat space $\mathscr{X}_r(q, d, p)$ is an intermediate theoretical device to shift from periodic polymers to aperiodic polymers like DNA and RNA in the Fukui Project. The space $\mathscr{X}_r(q, d, p)$ is a Banach algebra for all $1 \le p \le \infty$, and $\mathscr{X}_r(q, d, p)$ forms a C^*-algebra for $p = 2$. Here, polymer moiety size number q and dimension number d are arbitrarily given positive integers. The generalized repeat space $\mathscr{X}_r(q, d)$ is contained in the normed repeat space $\mathscr{X}_r(q, d, p)$, which in turn is contained in one of its super spaces $\mathscr{X}_B(q, d, p)$ so that aperiodic polymers can be represented and investigated within this super space $\mathscr{X}_B(q, d, p)$.

The normed repeat space $\mathscr{X}_r(q, d, p)$ and its super space $\mathscr{X}_B(q, d, p)$ are most fundamental in the second generation Fukui project. In the present article, keeping the above mentioned Fukui's DNA problem in mind, we formulate a prototypal problem of the normed repeat space $\mathscr{X}_r(q, d, p)$ and its super space $\mathscr{X}_B(q, d, p)$ in Sect. 3 after introducing definitions of symbols in Sect. 2. The notions of normed repeat space $\mathscr{X}_r(q, d, p)$, its super space $\mathscr{X}_B(q, d, p)$, and the repeat space $\mathscr{X}_r(q, d)$ are reviewed in the appendix given after the last Sect. 4, where the affirmative solution of the prototypal problem I is given as Theorem I. The present paper also provides three "Challenging Problems" II–IV. These are called parallel problems in view of the structural parallelism observed when compared with the original problem I. The challenging problems, which would allow multiple approaches, are specially designed for those who are interested in interdisciplinary investigations. They are also designed for those who are interested in computer graphic art and/or mathematical pedagogy.

The reader is also referred to the series of article [4–7], entitled 'Proof of the Fukui conjecture via resolution of singularities and related methods'. In this series, we develop fundamental theoretical tools, using methods from the field of resolution of singularities and analytic curves. These tools are essential in structurally elucidating the assertion of the Fukui conjecture (concerning the additivity problems) and the crux of the functional asymptotic linearity theorem (functional ALT) that proves the conjecture in a broad context. This conjecture is a vital guideline for a future development of the repeat theory (RST) [1–18], which is the central unifying theory in the First and the Second Generation Fukui Project.

2 Definition of symbols

Throughout, let $\mathbb{Z}^+$, $\mathbb{Z}_0^+$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$, denote respectively the set of all positive integers, nonnegative integers, integers, real numbers, and complex numbers. For each positive integer n, $\boldsymbol{M}_n(\mathbb{R})$ denotes the set of all $n \times n$ real matrices, and $\boldsymbol{M}_n(\mathbb{C})$ denotes the set of all $n \times n$ complex matrices.

Let $I \subset \mathbb{R}$ denote a closed interval. The symbol $C(I)$ denotes the real Banach space of all real-valued continuous functions on I equipped with the norm given by

$$\|\varphi\| = \sup\{|\varphi(t)| : t \in I\}. \tag{2.1}$$

Let $P(I)$ denote the subspace of $C(I)$ of all polynomial functions with real coefficients. Note, by the Stone-Weierstrass Theorem, $P(I)$ is dense in $C(I)$.

Let J denote a compact subset of $\mathbb{C}$. The symbol $C(J)$ denotes the complex Banach space of all complex-valued continuous functions on J equipped with the norm given by

$$\|\varphi\| = \sup\{|\varphi(t)| : t \in J\}. \tag{2.2}$$

Let $P(J)$ denote the subspace of $C(J)$ of all polynomial functions in variables z and $\bar{z}$ with complex coefficients. Note, by the Stone-Weierstrass Theorem, $P(J)$ is dense in $C(J)$.

We recall here the notion of the 'function' $\varphi(M)$ of a normal matrix $M \in \boldsymbol{M}_n(\mathbb{C})$. Let

$$M = \mu_1 P_{(1)} + \cdots + \mu_r P_{(r)} \tag{2.3}$$

be the spectral resolution of the normal matrix M, where $\mu_1, \ldots, \mu_r$ are all the distinct eigenvalues of M and $P_{(1)}, \ldots, P_{(r)}$ are corresponding eigenprojections. Let J be a subset of $\mathbb{C}$ that contains all the eigenvalues of M and let φ be a complex-valued function defined on J. Then, we define $\varphi(M)$ by

$$\varphi(M) = \varphi(\mu_1) P_{(1)} + \cdots + \varphi(\mu_r) P_{(r)}. \tag{2.4}$$

The fact that it is well defined is easily seen by the uniqueness of the spectral resolution.

Let U be an $n \times n$ unitary matrix such that

$$M = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U^{-1} \tag{2.5}$$

where $\lambda_1, \ldots, \lambda_n$ are all the eigenvalues of M counted with multiplicity, then one gets

$$\varphi(M) = U \operatorname{diag}(\varphi(\lambda_1), \ldots, \varphi(\lambda_n)) U^{-1}. \tag{2.6}$$

For the definitions of the following symbols $\mathscr{X}_n(q,d)$, $\mathscr{X}_H(q,d)$, $\mathscr{X}_{nr}(q,d)$, and $\mathscr{X}_{Hr}(q,d)$, the reader is referred to the definitions of $\mathscr{X}(q,d)$ and $\mathscr{X}_r(q,d)$, which are given in the appendix of the present article.

Let $\{M_N\} \in \mathscr{X}_r(q,d)$. A subset J of $\mathbb{C}$ is said to be *compatible with* $\{M_N\}$ if all the eigenvalues of M_N are contained in the set J for all $N \in \mathbb{Z}^+$. The relation $\mathscr{X}_r(q,d) \subset \mathscr{X}_B(q,d,p)$ given in (B.9) of the appendix (or Proposition 4.8 in [15]) implies that for any $\{M_N\} \in \mathscr{X}_r(q,d)$ there exists a compact subset J of $\mathbb{C}$ compatible with $\{M_N\}$.

Let $(q, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and let

$$\mathscr{X}_n(q,d) := \{\{M_N\} \in \mathscr{X}(q,d) : {M_N}^* M_N = M_N {M_N}^* \text{ for all } N \in \mathbb{Z}^+\}, \quad (2.7)$$
$$\mathscr{X}_H(q,d) := \{\{M_N\} \in \mathscr{X}(q,d) : {M_N}^* = M_N \text{ for all } N \in \mathbb{Z}^+\}, \quad (2.8)$$
$$\mathscr{X}_{nr}(q,d) := \mathscr{X}_n(q,d) \cap \mathscr{X}_r(q,d), \quad (2.9)$$
$$\mathscr{X}_{Hr}(q,d) := \mathscr{X}_H(q,d) \cap \mathscr{X}_r(q,d). \quad (2.10)$$

The following symbol plays a dominant role in the formulation of a prototypal problem of the normed repeat space $\mathscr{X}_r(q,d,p)$ and its super space $\mathscr{X}_B(q,d,p)$ given in the next section:

$$\mathscr{X}_{\alpha\text{-exist}}(q,d,p) := \left\{\{M_N\} \in \mathscr{X}_B(q,d,p) : \lim_{N\to\infty}\left[(\mathrm{Tr}M_N)/N^d\right] \text{ exists in } \mathbb{C}\right\}. \quad (2.11)$$

The following symbols play an important role in the formulation of parallel problems given in the next section: Let $n \in \mathbb{Z}^+$. Let

$$I_n := \text{ the } n \times n \text{identity matrix,}$$

$$J_n := \begin{pmatrix} & & & & 1 \\ & \mathbf{0} & & 1 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & 1 & & \mathbf{0} & \\ 1 & & & & \end{pmatrix} \in \boldsymbol{M}_n(\mathbb{R}), \quad (2.12)$$

that is J_n is the $n \times n$ matrix defined by $(J_n)_{ab} = 1$ if $a+b = n+1$, and $(J_n)_{ab} = 0$ otherwise. Note that J_n is a real-symmetric matrix with $J_n^2 = I_n$.

If M is an $n \times n$ complex matrix, let M^0 denote I_n. We remark that if $\{M_N\} \in \mathscr{X}_r(q,d)$ then we have

$$\left\{M_N^0\right\} = \{I_{qN^d}\} \in \mathscr{X}_r(q,d), \quad (2.13)$$

which is easily verified by the definition of $\mathscr{X}_r(q,d)$.

3 A prototypal problem of the normed repeat space $\mathscr{X}_r(q,d,p)$ and its super space $\mathscr{X}_B(q,d,p)$, and parallel problems

Alpha existence theorems are among most important theorems in the repeat space theory. They are fundamental for the proof of the Fukui conjecture, which is a guiding conjecture in the First and Second Generation Fukui Project. We reproduce here the following typical alpha existence theorem, (cf. theorem 4.3 in [15]):

Theorem A (Normal alpha existence theorem). *Let $q, d \in \mathbb{Z}^+$. Let $\{M_N\} \in \mathscr{X}_{nr}(q, d)$, let $J \subset \mathbb{C}$ be a compact set compatible with $\{M_N\}$. Let $\varphi \in C(J)$, then there exists an $\alpha(\varphi) \in \mathbb{C}$ such that*

$$\mathrm{Tr}\varphi(M_N)/N^d \to \alpha(\varphi) \tag{3.1}$$

as $N \to \infty$.

Let $\{M_N\} \in \mathscr{X}_{nr}(q, d)$, let $J \subset \mathbb{C}$ be a compact set compatible with $\{M_N\}$. Let $\varphi \in C(J)$, then it is not difficult to see that

$$\{\varphi(M_N)\} \in \mathscr{X}_r(q, d, 2). \tag{3.2}$$

(Cf. Theorem I(i) in Sect. 4 for the validity of this statement.) So, to prove Theorem A above, it suffices to prove Proposition I in the following problem I, which is the aforementioned prototypal problem of the normed repeat space $\mathscr{X}_r(q, d, p)$ and its super space $\mathscr{X}_B(q, d, p)$:

Problem I Prove or disprove the following proposition:

Proposition I $\mathscr{X}_r(q, d, p) \subset \mathscr{X}_{\alpha\text{-exist}}(q, d, p)$.

The reader is invited to tackle the following set of three problems, problems II–IV, which are called parallel problems in view of the structural parallelism observed when compared with the original problem I. We remark that our solutions of these parallel problems have helped a great deal in the affirmative solution of problem I, which is going to be presented in Sect. 4.

Challenging Parallel Problem II Prove or disprove the following proposition:

Proposition II *Let $q, d \in \mathbb{Z}^+$.Let $\{M_N\} \in \mathscr{X}_{Hr}(q, d)$, let $I \subset \mathbb{R}$ with $0 \in I$ be a closed interval compatible with $\{M_N\}$.Let $\{L_N\} \in \mathscr{X}_r(q, d)$ and let $\mathbf{0}$ denote the zero element of $\mathscr{X}_r(q, d)$. Consider the following statements:*

(i) $\{L_N\}\{M_N\} = \mathbf{0}$,
(ii) $\{L_N\}\{\varphi(M_N)\} = \mathbf{0}$ for all $\varphi \in P(I)$ with $\varphi(0) = 0$,
(iii) $\{L_N\}\{\varphi(M_N)\} = \mathbf{0}$ for all $\varphi \in C(I)$ with $\varphi(0) = 0$.

Then, we have the following implications:

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Challenging Parallel Problem III Prove or disprove the following proposition:

Proposition III *Let $q \in \mathbb{Z}^+$. Let M be a $q \times q$ Hermitian matrix and let $I \subset \mathbb{R}$ with $0 \in I$ be a closed interval that contains all the eigenvalues of M. Let L be a $q \times q$ complex matrix and let $\mathbf{0}$ denote the $q \times q$ zero matrix. Consider the following statements:*

(i) $LM = \mathbf{0}$,
(ii) $L\varphi(M) = \mathbf{0}$ for all $\varphi \in P(I)$ with $\varphi(0) = 0$,
(iii) $L\varphi(M) = \mathbf{0}$ for all $\varphi \in C(I)$ with $\varphi(0) = 0$.

Then, we have the following implications:

(i) ⇒ (ii) ⇒ (iii).

Challenging Parallel Problem IV Prove or disprove the following proposition:

Proposition IV *Let* $q \in \mathbb{Z}^+$*. Let* M *be a* $q \times q$ *real-symmetric matrix and let* $I \subset \mathbb{R}$ *with* $0 \in I$ *be a closed interval that contains all the eigenvalues of* M*. Let* L *be a* $q \times q$ *real matrix defined by*

$$L = J_q - I_q, \tag{3.3}$$

where J_q *and* I_q *are defined in Sect. 2. Let* $\mathbf{0}$ *denote the* $q \times q$ *zero matrix. Consider the following statements:*

(i) $LM = \mathbf{0}$,
(ii) $L\varphi(M) = \mathbf{0}$ for all $\varphi \in P(I)$ with $\varphi(0) = 0$,
(iii) $L\varphi(M) = \mathbf{0}$ for all $\varphi \in C(I)$ with $\varphi(0) = 0$.

Then, we have the following implications:

(i) ⇒ (ii) ⇒ (iii).

The second generation Fukui project is an international, interdisciplinary, and intergenerational project. The Matrix Interface Program, Logical Interface Program, Matrix Art Program, and Math Art Program have been playing an important role in the project, especially for the project associates of younger generation.

The Matrix Interface Program is closely associated with the philosophy and applications of the 'logical interface' exploited in the first generation Fukui project. (Cf. [18] for details.) The present author recalls with pleasure discussions on the additivity and reactivity problems of organic compounds with the late Professors Kenichi Fukui and Haruo Shingu, especially those valuable discussions on the *dialectic and mutually beneficial interplay between theory and experiment*, which later lead the author to form the notion of the 'logical interface'. In the Matrix Interface Program, the present author has devised a new notion of the power-ω-energy $E^{\omega}(M)$ of any $n \times n$ normal matrix as follows: Let ω be a positive real number, let M be an $n \times n$ normal matrix, let

$$E^{\omega}(M) := \operatorname{Tr} \varphi_{\omega}(M), \tag{3.4}$$

where $\varphi_{\omega} : \mathbb{C} \to \mathbb{R}$ is defined by $\varphi_{\omega}(t) = |t|^{\omega}$. The power-1/2-energy of a normal matrix M is called the zero-point energy (or, zero energy, for short) of matrix M. The power-1-energy of a normal matrix is called the pi energy of matrix M. (*Note*: If M denotes an adjacency matrix of graph G, we call $E^{\omega}(M)$ the power-ω-energy of graph G.) The notion of the power-ω-energy $E^{\omega}(M)$ is especially useful for those who wish to efficiently learn quantum chemical aspects in the Fukui project. It is also instructive to consider the function-φ-energy $E^{\varphi}(M)$ defined by $E^{\varphi}(M) := \operatorname{Tr} \varphi(M)$ for any $n \times n$ normal matrix, where $\varphi : \mathbb{C} \to \mathbb{R}$ is a fixed function, or function φ is given from a certain functional space. The notion of the function-φ-energy $E^{\varphi}(M)$ is especially useful for those who are interested in statistical thermodynamic data of organic compounds and/or in methods of functional analysis used in the Fukui project.

The Matrix and Math Art Programs are a philosophical and methodical extension, from science towards art, of Fukui's approach and also of the Approach via the Aspect of Form and General Topology (cf. [16] and references therein) in the repeat space theory, which is the central unifying theory in the first and second generation Fukui project. In the Matrix Art Program, the present author has created computer programs in MATLAB for a set of experiments in computer graphic art, using fundamental methods of the repeat space theory and also referring to the above parallel problems II–IV.

In Proposition IV, set $q = 4$,

$$M = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 2 \\ 2 & 3 & 3 & 2 \\ 1 & 2 & 2 & 1 \end{pmatrix}, \tag{3.5}$$

and observe that $J_4 M = M$ so that $LM = \mathbf{0}$. Taking the transpose of both sides, we also have $M J_4 = M$ so that $ML = \mathbf{0}$. This means that the matrix M has the horizontal and vertical mirror image symmetry. By computer experiments using MATLAB, one can visually recognize the same mirror image symmetry in the contour map of $\varphi(M)$, for various $\varphi \in P(I)$ with $\varphi(0) = 0$ and for various $\varphi \in C(I)$ with $\varphi(0) = 0$. The following Fig. 1 shows a 3D map and a 2D contour map of a 200×200 real symmetric matrix M with $J_{200} M = M$ and $M J_{200} = M$. (The matrix M given here provides an approximate presentation of the 2 dimensional Takagi function, which is also referred to as Magic Mountain in our Matrix Art Program.)

The following Fig. 2 shows a contour map of $\varphi(M)$ with $\varphi(t) = \sin t$.

Remark The above pictures M and $\varphi(M)$ and numerous similar pictures have been obtained by the present author and members of the Fukui project association in Tsuyama National College of Technology. Special thanks are due to T. Fukuda and other members of the Fukui project association, who contributed to the Matrix Interface, Logical Interface, Matrix Art, and Math Art Programs.

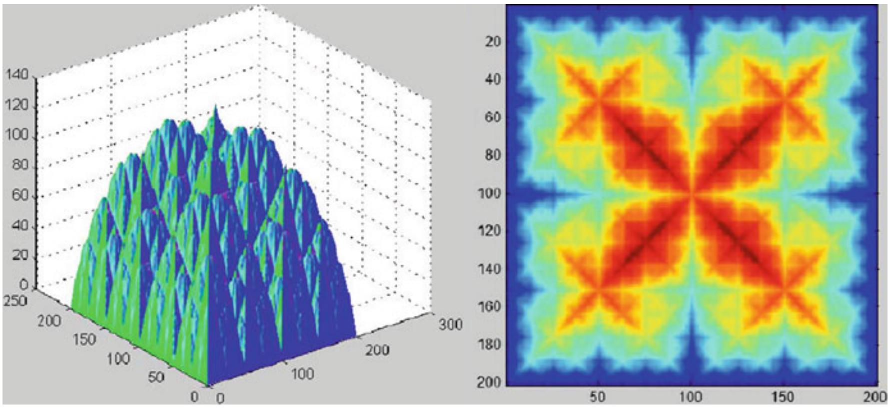

Fig. 1 Magic Mountain$_{200}$

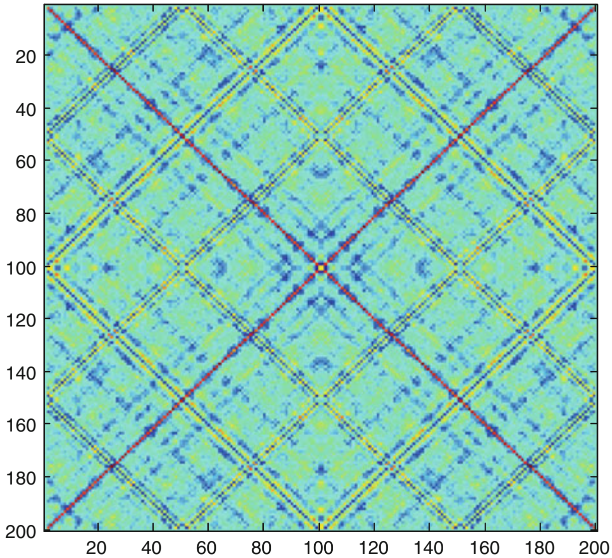

Fig. 2 Contour map of sin(Magic Mountain$_{200}$)

4 Solution of the prototypal problem

In this section, we provide an affirmative solution of the prototypal problem I as Theorem I. Before proceeding further, the reader is asked to briefly review Theorem B1 in the appendix, which asserts $\mathscr{X}_r(q,d) \subset \mathscr{X}_B(q,d,p)$, and the definitions of the normed repeat space $\mathscr{X}_r(q,d,p)$ and its related notions in the appendix.

Theorem I *Let $q, d \in \mathbb{Z}^+$ and let $1 \leq p \leq \infty$. Let $\{M_N\} \in \mathscr{X}_{nr}(q,d)$, let $J \subset \mathbb{C}$ be a compact set compatible with $\{M_N\}$. Let $\varphi \in C(J)$, then we have*

(i) $\{\varphi(M_N)\} \in \mathscr{X}_r(q,d,2)$,
(ii) $\mathscr{X}_{\alpha\text{-exist}}(q,d,p)$ is *a closed linear subspace of* $\mathscr{X}_B(q,d,p)$,
(iii) $\mathscr{X}_r(q,d,p) \subset \mathscr{X}_{\alpha\text{-exist}}(q,d,p)$,
(iv) $\{\varphi(M_N)\} \in \mathscr{X}_{\alpha\text{-exist}}(q,d,2)$.

Proof

(i) Under the assumption of the theorem, suppose that $\phi \in P(J)$ is such that $\phi(\lambda) = \sum_{i,j=0}^{n} a_{ij}\lambda^i\bar{\lambda}^j$ with $a_{ij} \in \mathbb{C}$, for $0 \leq i, j \leq n$, then $\phi(M_N) = \sum_{i,j=0}^{n} a_{ij} {M_N}^i ({M_N}^*)^j$.
Let $\varphi_k \in P(J)$ be such that $\varphi_k \to \varphi$ in $C(J)$. (The complex Stone-Weierstrass theorem implies the existence of such a sequence φ_k.) Since $\mathscr{X}_r(q,d)$ is a $*$-algebra by Theorem A1 in the appendix and since $\left\{M_N^0\right\} \in \mathscr{X}_r(q,d)$ by (2.13), we infer that

$$\{\varphi_k(M_N)\} \in \mathscr{X}_r(q,d), \tag{4.1}$$

which is contained in $\mathscr{X}_B(q,d,2)$ by (B.9) in the appendix.

Note that

$$\begin{aligned}&\|\{\varphi(M_N)\} - \{\varphi_k(M_N)\}\|_2\\&\quad = \sup\{\|\varphi(M_N) - \varphi_k(M_N)\|_2 : N \geq 1\} \leq \|\varphi - \varphi_k\| \to 0 \quad (4.2)\end{aligned}$$

as $k \to \infty$. Thus, we see that $\{\varphi(M_N)\}$ is an element of the closure of $\mathscr{X}_r(q, d)$. By the definition of $\mathscr{X}_r(q, d, 2)$, we have $\{\varphi(M_N)\} \in \mathscr{X}_r(q, d, 2)$.

(ii) Since $\mathscr{X}_{\alpha\text{-exist}}(q, d, p)$ is obviously a linear subspace of $\mathscr{X}_B(q, d, p)$, we prove that it is closed. Let l^∞ denote the normed space of all the sequences $\{a_N\}$ in $\mathbb{C}$ with $\sup\{|a_N| : N \geq 1\} < \infty$, equipped with the norm given by

$$\|\{a_N\}\| = \sup\{|a_N| : N \geq 1\}. \quad (4.3)$$

Define $\theta_1 : \mathscr{X}_B(q, d, p) \to l^\infty$ and $\theta_2 : l^\infty \to \mathbb{R}$ by

$$\theta_1(\{M_N\}) = \left\{(\mathrm{Tr}\ M_N)/N^d\right\}, \quad (4.4)$$

$$\theta_2(\{a_n\}) = \lim_{n_0\to\infty} \sup\{|a_m - a_n| : m,\ n \geq n_0\}. \quad (4.5)$$

By lemma 3.2 in [8], we know that θ_2 is continuous. Since

$$\mathscr{X}_{\alpha\text{-exist}}(q, d, p) = (\theta_2 \circ \theta_1)^{-1}(\{0\}), \quad (4.6)$$

for the proof of (ii), it remains to show that θ_1 is well defined and continuous. Let $\{M_N\} \in \mathscr{X}_B(q, d, p)$. Let $\lambda_i(M_N)$, with $i = 1, 2, \ldots, qN^d$, denote all the qN^d eigenvalues of M_N, counted with multiplicity. First, note that if λ is an eigenvalue of M_N, then $|\lambda| \leq \|M_N\|_p$. Second note that

$$|\mathrm{Tr}\ M_N| = \left|\sum_{i=1}^{qN^d} \lambda_i(M_N)\right| \leq \sum_{i=1}^{qN^d} |\lambda_i(M_N)| \leq qN^d \|M_N\|_p, \quad (4.7)$$

which shows that θ_1 is well defined, hence

$$\begin{aligned}\|\theta_1(\{M_N\})\| &= \sup\{|\mathrm{Tr}\ M_N|/N^d : N \geq 1\}\\&\leq q \sup\left\{\|M_N\|_p : N \geq 1\right\}\\&= q\|\{M_N\}\|_p, \quad (4.8)\end{aligned}$$

which implies that the linear operator θ_1 is bounded hence continuous.

(iii) We know from lemma 4.1 (1) in [15] that

$$\mathscr{X}_r(q, d) \subset \mathscr{X}_{\alpha\text{-exist}}(q, d, p). \quad (4.9)$$

By the definition of $\mathscr{X}_r(q, d, p)$ and (ii), the conclusion follows.

(iv) The conclusion directly follows from (i) and (iii). □

Thus, our problem I has been affirmatively solved.

We remark that the notion of the normed repeat space reviewed in the appendix unites the approaches via the aspects of form and general topology exploited in a variety of asymptotic analyses of molecular networks in [1–18] and references therein. Equipped with the machinery of Banach algebras and C^*-algebras, the notion of normed repeat space with the above-mentioned new unifying power forms a basis of the second generation Fukui project. For a review of the first generation Fukui project, whose basic philosophy we would like to carry on to the second generation project, the reader is referred to Ref. [2] entitled 'Note on the repeat space theory – its development and communications with Prof. Kenichi Fukui'.

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Appendix

Review of the generalized repeat space and the normed repeat space

Generalized repeat space

There are several equivalent ways of defining the generalized repeat space $\mathscr{X}_r(q, d)$ with a given size $(q, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$. We shall recall below the definition that uses the notion of the sum of subspaces of a linear space (cf. Refs. [1,13,15,16]).

Fix $(q, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and let $\mathscr{X}(q, d)$ denote the set of all matrix sequences whose Nth term M_N is an arbitrary $qN^d \times qN^d$ complex matrix, $N \in \mathbb{Z}^+$. This set constitutes a $*$-algebra over the field $\mathbb{C}$ with term-wise addition, scalar multiplication, multiplication

$$\{M_N\} + \{M_N{}'\} = \{M_N + M_N{}'\}, \tag{A.1}$$

$$k\{M_N\} = \{kM_N\}, \tag{A.2}$$

$$\{M_N\}\{M_N{}'\} = \{M_N M_N{}'\}, \tag{A.3}$$

and involution $(.)^* : \mathscr{X}(q, d) \to \mathscr{X}(q, d)$ defined by

$$\{M_N\}^* = \left\{M_N{}^*\right\}, \tag{A.4}$$

where the * on the right-hand side of (A.4) denotes the adjoint operation.

Let P_N denote an $N \times N$ real-orthogonal matrix given by

$$P_N = \begin{pmatrix} 0 & 1 & & & & & \\ & 0 & 1 & & & \mathbf{0} & \\ & & 0 & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & 0 & 1 & \\ & \mathbf{0} & & & & 0 & 1 \\ 1 & & & & & & 0 \end{pmatrix}.$$

Let ${P_N}^a := ({P_N}^{-1})^{-a}$ where $a \in \{-2, -3, \ldots\}$. (Note that ${P_N}^a$ equals the transpose of ${P_N}^{-a}$.)

Let S_N denote an $N \times N$ real idempotent matrix given by

$$S_N = \begin{pmatrix} 1 & 0 & & & & & \\ 0 & 0 & 0 & & & \mathbf{0} & \\ & 0 & 0 & \ddots & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \ddots & \ddots & 0 & \\ & \mathbf{0} & & & 0 & 0 & 0 \\ & & & & & 0 & 0 \end{pmatrix}.$$

(Note that S_N is idempotent so that ${S_N}^j = S_N$ whenever j is a strictly positive integer.)

Let ${P_N}^n$ denote the $N^d \times N^d$ matrix given by

$${P_N}^n = {P_N}^{n_1} \otimes {P_N}^{n_2} \otimes \cdots \otimes {P_N}^{n_d}, \tag{A.5}$$

where $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$, and $\otimes$ denotes the Kronecker product.

Let ${S_N}^k$ denote the $N^d \times N^d$ matrix given by

$${S_N}^k = {S_N}^{k_1} \otimes {S_N}^{k_2} \otimes \cdots \otimes {S_N}^{k_d}, \tag{A.6}$$

where $k = (k_1, k_2, \ldots, k_d) \in (\mathbb{Z}^+ \cup \{0\})^d$.

Let $\mathscr{V}^k(q, d)$ with $k = (k_1, k_2, \ldots, k_d) \in \{0, 1\}^d$ denote the subset of $\mathscr{X}(q, d)$ defined by

$$\mathscr{V}^k(q, d) = \{\{M_N\} \in \mathscr{X}(q, d) : \exists m, n \in \mathbb{Z}^d, \exists Q \in \boldsymbol{M}_q(\mathbb{C}) \text{ such that } M_N = ({P_N}^m {S_N}^k {P_N}^n) \otimes Q \text{ for all } N \gg 0\}. \tag{A.7}$$

Let span $\mathscr{V}^k(q, d)$ with $k = (k_1, k_2, \ldots, k_d) \in \{0, 1\}^d$ denote the linear span of $\mathscr{V}^k(q, d)$.

We defined three fundamental linear subspaces

$\mathscr{X}_r(q,d)$, $\mathscr{X}_\alpha(q,d)$, and $\mathscr{X}_\beta(q,d)$ of $\mathscr{X}(q,d)$ by

$$\mathscr{X}_r(q,d) = \sum_{k\in\{0,1\}^d} \text{span } \mathscr{V}^k(q,d), \tag{A.8}$$

$$\mathscr{X}_\alpha(q,d) = \text{span } \mathscr{V}^{\mathbf{0}}(q,d), \tag{A.9}$$

$$\text{where } \mathbf{0} = (0,0,\ldots,0) \in \{0,1\}^d\ , \tag{A.10}$$

$$\mathscr{X}_\beta(q,d) = \sum_{k\in\{0,1\}^d\setminus\{\mathbf{0}\}} \text{span } \mathscr{V}^k(q,d). \tag{A.11}$$

In (A.8) and (A.11), the Σ denotes the sum of subspaces in the obvious manner.

We call $\mathscr{X}_r(q,d)$, $\mathscr{X}_\alpha(q,d)$, $\mathscr{X}_\beta(q,d)$, respectively, the generalized repeat space, generalized alpha space, and generalized beta space with size (q,d), and each element of $\mathscr{X}_r(q,d)$, $\mathscr{X}_\alpha(q,d)$, $\mathscr{X}_\beta(q,d)$, respectively, a generalized repeat sequence, generalized alpha sequence, and generalized beta sequence with size (q,d).

The following is one of the most fundamental theorems in the repeat space theory.

Theorem A1 *For all $q, d \in \mathbb{Z}^+$, $\mathscr{X}_r(q,d)$ forms a $*$-algebra.*

Proof This was proved in Ref. [15]. □

For the special definition of the generalized repeat space with size $(q,1)$, set $d = 1$ in the definition of $\mathscr{V}^k(q,d)$ given by (A.7) and observe that

$$\begin{aligned}\mathscr{X}_\alpha(q,1) &= \text{span } \mathscr{V}^0(q,1)\\ &= \text{span } \{\{M_N\} \in \mathscr{X}(q,1) : \exists\, m \in \mathbb{Z}, \exists\, Q \in \boldsymbol{M}_q(\mathbb{C}) \text{ such that}\\ &\qquad M_N = {P_N}^m \otimes Q \quad \text{for all } N \gg 0\},\end{aligned} \tag{A.12}$$

$$\begin{aligned}\mathscr{X}_\beta(q,1) &= \text{span } \mathscr{V}^1(q,1)\\ &= \text{span } \{\{M_N\} \in \mathscr{X}(q,1) : \exists\, m,\ n \in \mathbb{Z}, \exists\, Q \in \boldsymbol{M}_q(\mathbb{C}) \text{ such that}\\ &\qquad M_N = ({P_N}^m S_N {P_N}^n) \otimes Q \quad \text{for all } N \gg 0\},\end{aligned} \tag{A.13}$$

and note that

$$\mathscr{X}_r(q,1) = \mathscr{X}_\alpha(q,1) + \mathscr{X}_\beta(q,1). \tag{A.14}$$

Normed repeat space

Let $\boldsymbol{C}^n$ denote the set of all column n-vectors. For each $1 \le p < \infty$, let

$$\|\xi\|_p := \|(\xi_1,\ldots,\xi_n)^T\|_p = (|\xi_1|^p + \cdots + |\xi_n|^p)^{1/p}. \tag{B.1}$$

Let

$$\|\xi\|_\infty = \|(\xi_1,\ldots,\xi_n)^T\|_\infty = \max\{|\xi_i| : 1 \le i \le n\}. \tag{B.2}$$

For each positive integer n and $1 \le p \le \infty$, let Mat(n, p) denote the set of all $n \times n$ complex matrices with the norm given by

$$\|A\|_p = \sup\{\|Ax\|_p/\|x\|_p : x \in \boldsymbol{C}^n\backslash\{\mathbf{0}\}\}. \tag{B.3}$$

Fix $(q, d) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ and let $\mathscr{X}(q, d)$ denote the set of all matrix sequences whose Nth term M_N is an arbitrary $qN^d \times qN^d$ complex matrix, $N \in \mathbb{Z}^+$. This set constitutes a $*$-algebra over the field $\mathbb{C}$ with term-wise addition, scalar multiplication, multiplication

$$\{M_N\} + \{M_N{}'\} = \{M_N + M_N{}'\}, \tag{B.4}$$
$$k\{M_N\} = \{kM_N\}, \tag{B.5}$$
$$\{M_N\}\{M_N{}'\} = \{M_N M_N{}'\}, \tag{B.6}$$

and involution $(.)^* : \mathscr{X}(q, d) \to \mathscr{X}(q, d)$ defined by

$$\{M_N\}^* = \left\{M_N{}^*\right\}, \tag{B.7}$$

where the $*$ on the right-hand side of (B.7) denotes the adjoint operation.

For each $q, d \in \mathbb{Z}^+$ and $1 \le p \le \infty$, let

$$\mathscr{X}_B(q, d, p) := \left\{\{M_N\} \in \prod_{N=1}^{\infty} \text{Mat}\left(qN^d, p\right) : \|\{M_N\}\|_p := \sup_N \|M_N\|_p < \infty\right\}. \tag{B.8}$$

Note that $\mathscr{X}_B(q, d, p)$ forms a subalgebra of $\mathscr{X}(q, d)$. We also note that $\mathscr{X}_B(q, d, p)$ forms a Banach algebra for each $1 \le p \le \infty$ and a C^*-algebra for $p = 2$. The set $\mathscr{X}_B(q, d, p)$ is called the *bounded underlying space* (or *B-space* for short) *of type* (q, d, p).

Now recall the definition of the generalized repeat space with size (q, d), which is denoted by $\mathscr{X}_r(q, d)$ in (A.8).

Theorem B1 *For each* $q, d \in \mathbb{Z}^+$ *and* $1 \le p \le \infty$*, we have*

$$\mathscr{X}_r(q, d) \subset \mathscr{X}_B(q, d, p). \tag{B.9}$$

Proof This was proved in Ref. [3]. □

Definition of the Normed Repeat Space For each $q, d \in \mathbb{Z}^+$ and $1 \le p \le \infty$, let

$$\mathscr{X}_r(q, d, p) := \text{closure of } \mathscr{X}_r(q, d) \subset \mathscr{X}_B(q, d, p). \tag{B.10}$$

The set $\mathscr{X}_r(q, d, p)$ is called the *normed repeat space of type* (q, d, p).

Note that $\mathscr{X}_r(q, d, p)$ forms a Banach algebra for each $1 \le p \le \infty$ and a C^*-algebra for $p = 2$. This fact easily follows from the observation that linear operations, multiplication, and involution are all continuous operations and that any closed set in a complete metric space forms a complete metric subspace. (The reader is referred e.g. to refs. [19,20] for the fundamental properties of Banach algebras and C^*-algebras.)

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